Second case study: Network Creation Games (a.k.a. Local Connection Games)

Introduction

- Introduced in [FLMPS,PODC'03]
- A **Local Connection Game** (LCG) is a game that models the *ex-novo* creation of a network
- Players are nodes that:
  - Incur a cost for the (adjacent) links they personally activate;
  - Benefit from having the other nodes on the network as close as possible, in terms of length of shortest paths on the created network (notice they can use all the activated edges)

[FLMPS,PODC'03]: A. Fabrikant, A. Luthra, E. Maneva, C.H. Papadimitriou, S. Shenker, On a network creation game, PODC'03

The formal model

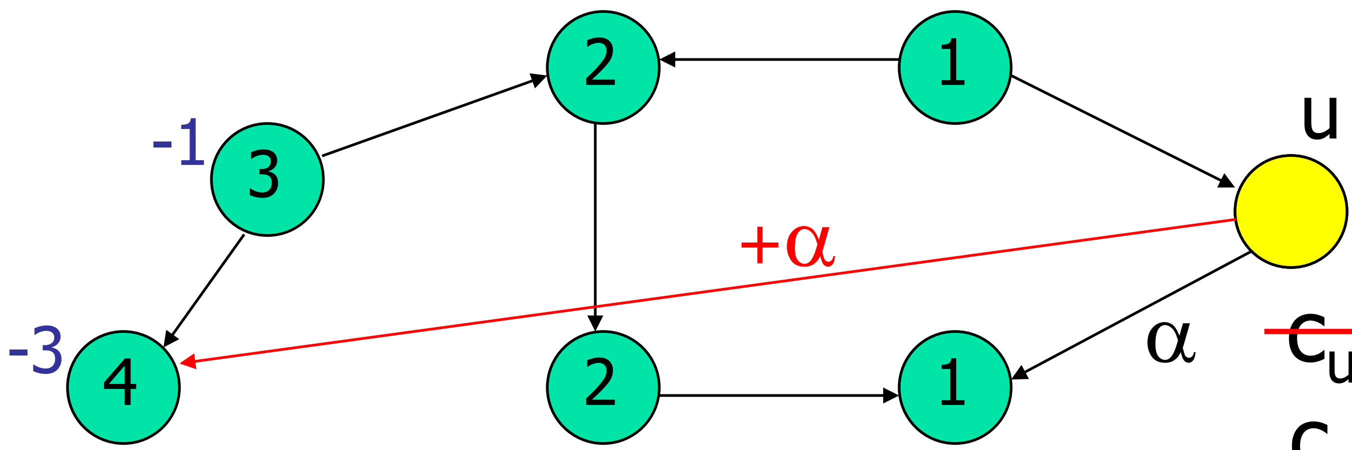
- n players: nodes $V=\{1,\dots,n\}$ in a graph to be built
- Strategy for player u: a set of incident edges (intuitively, a player buys these edges, that will be then used bidirectionally by everybody; however, only the owner of an edge can remove it, in case he decides to change his strategy)
- Given a strategy vector $S=(s_1,\dots, s_n)$, the constructed network will be the undirected graph $G(S)$
- player u's goal:
  - to spend as little as possible for buying edges (building cost)
  - to make the distance to other nodes as small as possible (usage cost)

The model

- Each edge has a real-value cost $\alpha \geq 0$
- $dist_{G(S)}(u,v)$: length of a shortest path (in terms of number of edges) in $G(S)$ between u and v
- n_u: number of edges bought by node u
- Player u aims to minimize its cost:

$$cost_u(S) = \alpha n_u + \sum_{v \in V} dist_{G(S)}(u,v)$$

Cost of a player: an example



~~$c_u=\alpha+13$~~

$c_u=2\alpha+9$

Convention: arrow *from* the node buying the link

Notice that if $\alpha<4$ this is an improving move for u

The social-choice function

- To evaluate the overall quality of a network, once again we consider the *utilitarian social cost*, i.e., the sum of all players' costs. Observe that:
  1. In $G(S)$ each term $dist_{G(S)}(u,v)$ contributes to the overall cost twice
  2. Each edge (u,v) is bough at most by one player

Social cost of a network $G(S)=(V,E)$:

$$SC(G(S))=\alpha|E| + \Sigma_{u,v\in V}\, dist_{G(S)}(u,v)$$

Some (bad) computational aspects of LCG

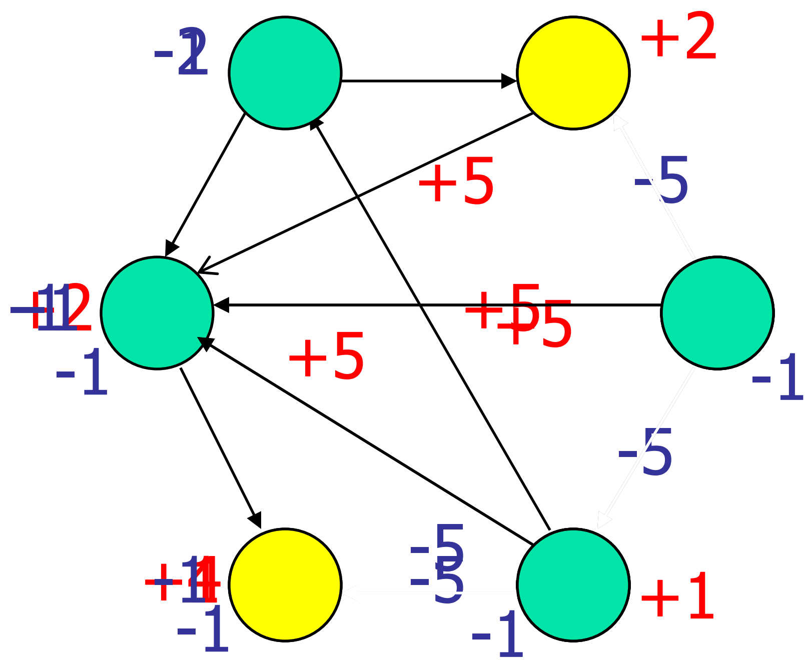
- LCG are not potential games (differently from GCG); this can be shown by providing an instance in which a sequence of improving moves will generate a cycle in the corresponding space of strategy profiles
- Computing a best-response move for a player is NP-hard (differently from GCG)
- The complexity of establishing the existence of an improving move for a player (decision problem) is open
- The complexity of establishing the existence of a NE for a given α (decision problem) is open

Our goal

- We use Nash equilibrium (NE) as the solution concept: Given a strategy profile S, the formed network G(S)=(V,E) is stable (for the given value α) if S is a NE
- Conversely, given a graph G=(V,E), it is stable if there exists a strategy vector S such that G=G(S), and S is a NE
- Observe that any stable network must be connected, since the distance between two nodes is infinite whenever they are not connected
- A network is *optimal* or *socially efficient* if it minimizes the social cost
- We aim to characterize the efficiency loss resulting from selfishness, by bounding the Price of Stability (PoS) and the Price of Anarchy (PoA)

Stable networks: an example

- Set α=5, and consider:



That's a stable network!

Theorem 1

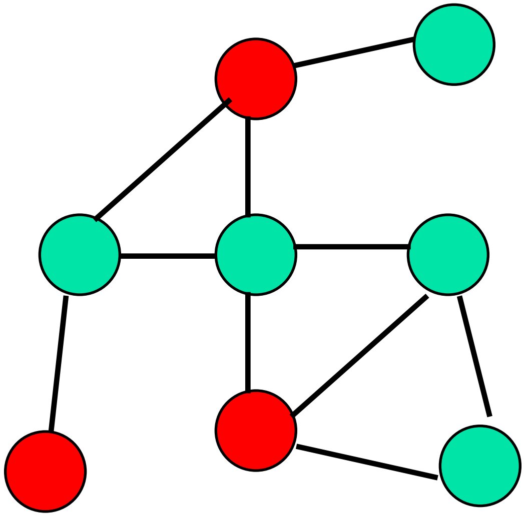
It is NP-hard, given the strategies of the other agents, to compute the best response of a given player in a LCG.

proof

Reduction from dominating set problem

Dominating Set (DS) problem

- Input:
  - a graph G=(V,E)
- Solution:
  - U⊆V, such that for every v∈V-U, there is u∈U with (u,v)∈E
- Measure:
  - Cardinality of U



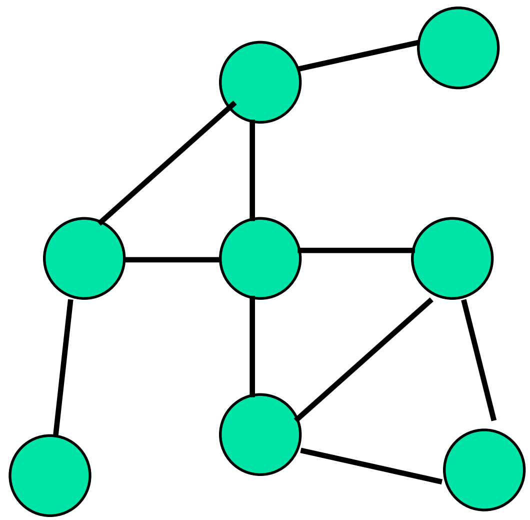
the reduction

$1<\alpha<2$

player i

Instance of MDS

$\Rightarrow$ Instance of LCG

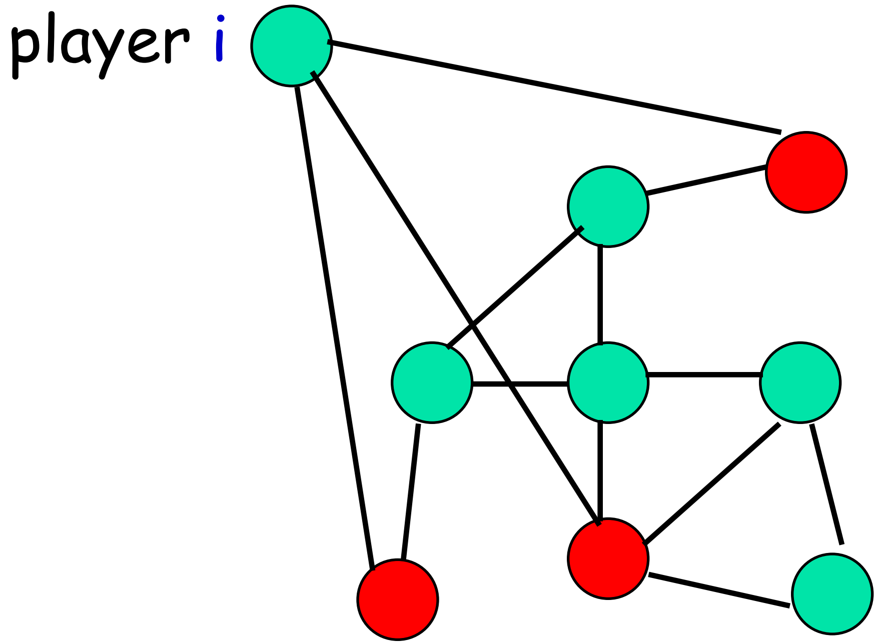


$G=(V,E) = G(S_{-i})$

We will show that player i has a strategy yielding a cost $\leq \alpha k+2n-k$ if and only if there is a DS of size $\leq$ k

the reduction

$1<\alpha<2$



$G=(V,E) = G(S_{-i})$

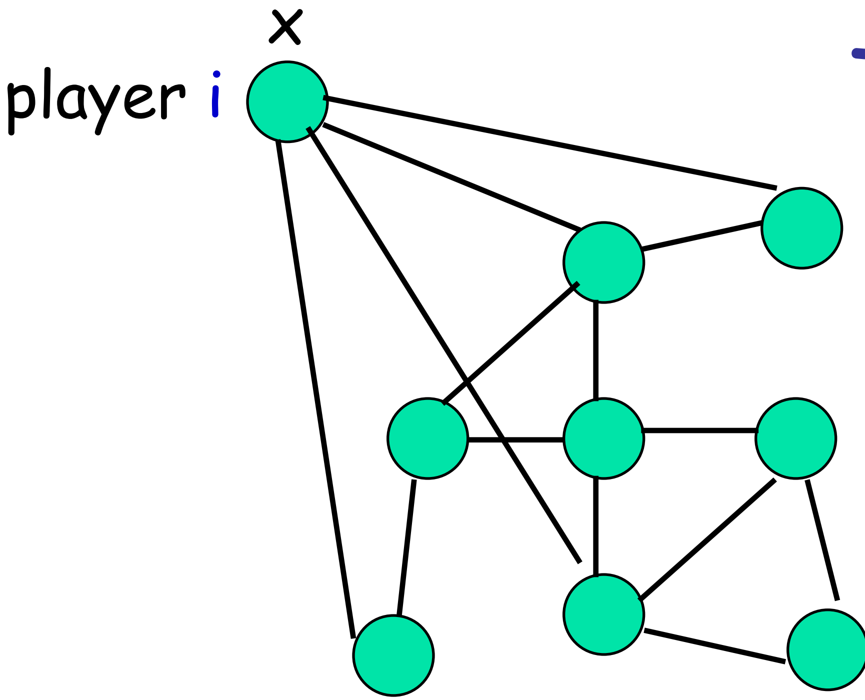
($\Leftarrow$)

easy: given a dominating set U of size at most k in G, we want to show that there exists a strategy for player i costing at most $\alpha k+2n-k$; then, let i buy edges incident to the nodes in U

⇒ Cost for i is $\alpha|U|+2n-|U| \leq (\alpha-1)\,|U| +2n$ which is maximum for $|U|=k$, since $1<\alpha<2$

$1<\alpha<2$

the reduction



$G=(V,E) = G(S_{-i})$

($\Rightarrow$)

Let S_i be a strategy giving to player i a cost $\leq \alpha k+2n-k$

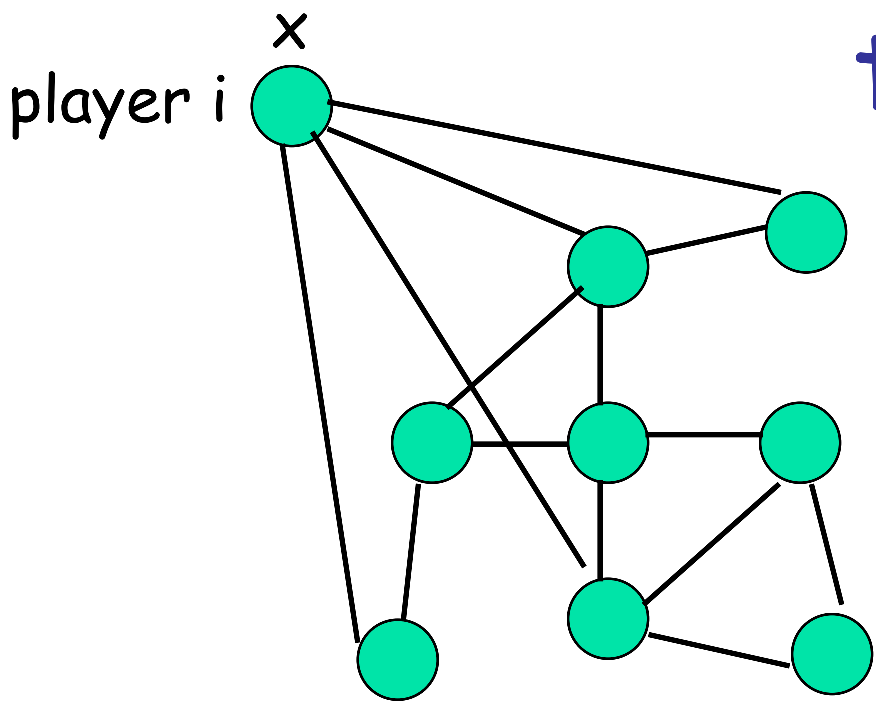
Modify S_i as follows:

repeat:

if there is a node v with distance ≥ 3 from x in G(S), then add edge (x,v) to S_i (this decreases the cost of player i)

$1<\alpha<2$

the reduction



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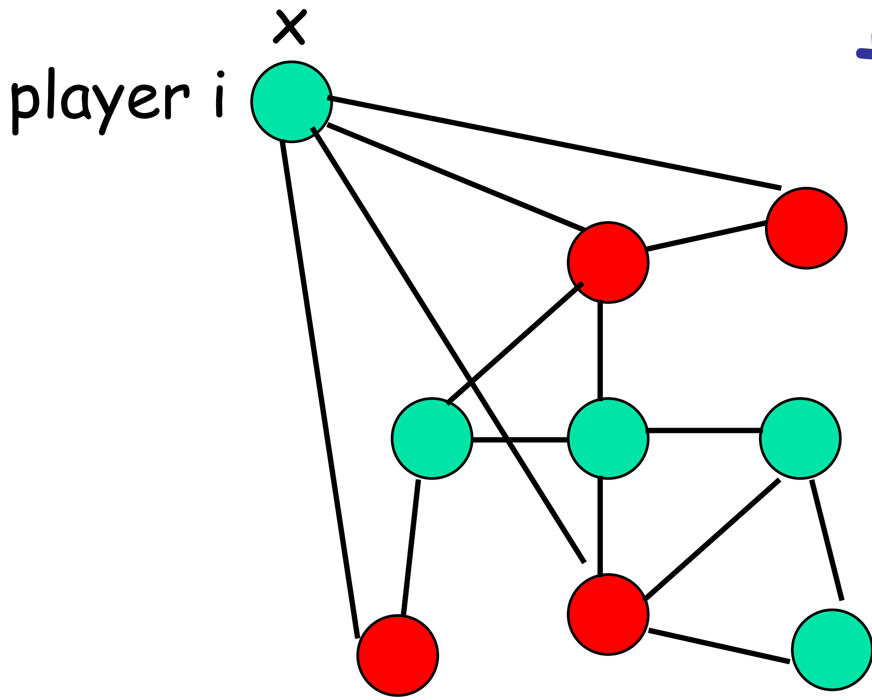
Finally, every node has distance either 1 or 2 from x

$1<\alpha<2$

the reduction



$G=(V,E) = G(S_{-i})$

($\Rightarrow$)

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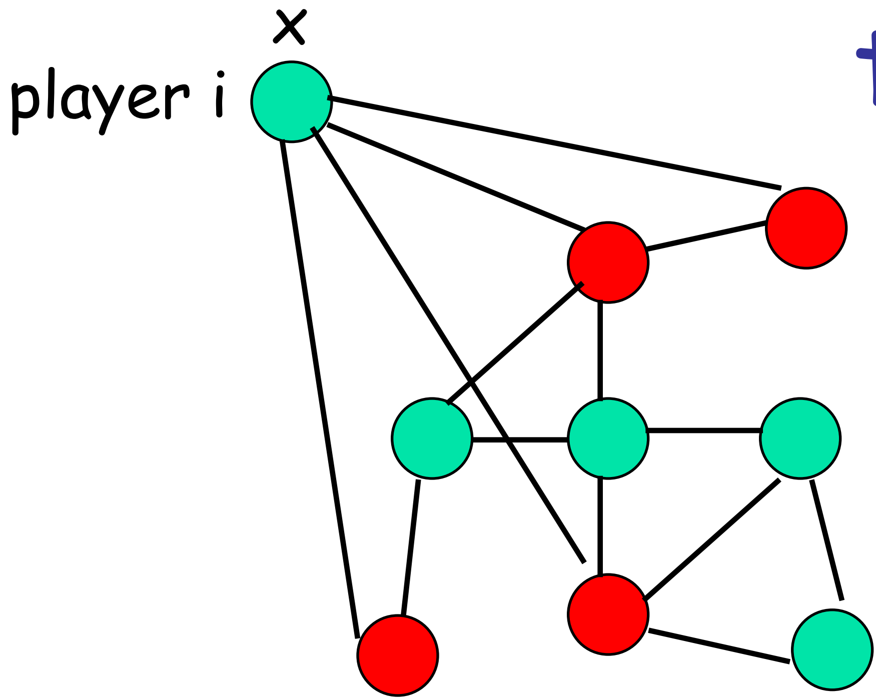
if there is a node v such with distance ≥3 from x in G(S), then add edge (x,v) to S_i (this decreases the cost)

Finally, every node has distance either 1 or 2 from x

Let U be the set of nodes at distance 1 from x…

$1<\alpha<2$

the reduction



$G=(V,E) = G(S_{-i})$

($\Rightarrow$)

...it is easy to see that U is a dominating set of the original graph G

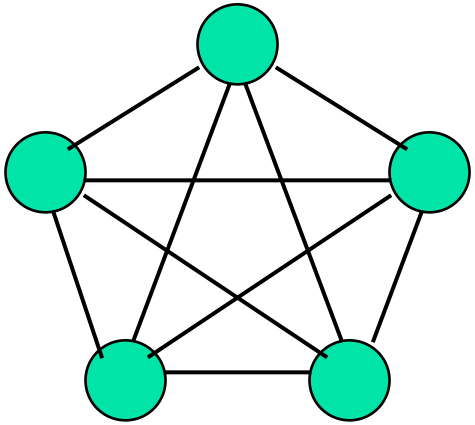
We have $cost_i(S)= \alpha|U|+2n-|U| \leq \alpha k+2n-k$

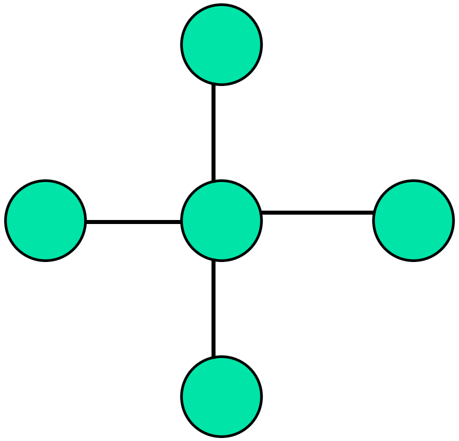
$(\alpha-1)|U| \leq (\alpha-1)k$ and since $\alpha>1$
$|U| \leq k$

How does an optimal network look like?

Some notation

K_n: complete graph
with n nodes





A star is a tree
with height at most 1
(when rooted at its
center)

Lemma 1

Il $\alpha \leq 2$ then the complete graph is an optimal solution, while if $\alpha \geq 2$ then the star is an optimal solution.

proof

Let $G=(V,E)$ be an optimal solution; $|E|=m$ and $SC(G)=OPT$

$$OPT = \alpha|E| + \Sigma_{u,v\in V}\, dist_G(u,v) \geq \alpha m + 2m \quad + \quad 2(n(n-1) - 2m)$$

$$=(\alpha-2)m + 2n(n-1)$$ ⟸ LB(m)

adjacent nodes at distance 1 (for $2m$)

non-adjacent pairs of nodes at distance ≥ 2 (for $2(n(n-1) - 2m)$)

Notice: LB(m) is equal to $SC(K_n)$ when $m=n(n-1)/2$, and to SC(star) when $m=n-1$; indeed:

$$SC(K_n) = \alpha\, n(n-1)/2 + n(n-1)$$

$$SC(star) = \alpha\,(n-1) + 2(n-1) + 2(n-1)(n-2) = \alpha\,(n-1) + 2(n-1)^2$$

and it is easy to see that they correspond to $LB(n(n-1)/2)$ and to $LB(n-1)$, respectively.

Proof (continued)

$G=(V,E)$: optimal solution;
$|E|=m$ and $SC(G)=OPT$

$LB(m)=(\alpha-2)m + 2n(n-1)$

$OPT \geq LB(m) \geq$

$LB(n-1) = SC(star)$

$\alpha \geq 2 \Rightarrow \alpha-2 \geq 0$ and so the cheapest network is obtained when m is small, i.e., for $m=n-1$

$\alpha \leq 2 \Rightarrow \alpha-2 \leq 0$ and so the cheapest network is obtained when m is large, i.e., for complete graphs

$LB(n(n-1)/2) = SC(K_n)$

■

Are complete graphs and stars stable?

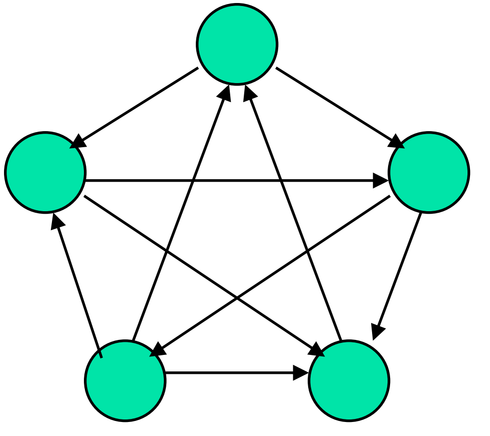
Lemma 2

Il $\alpha \leq 1$ the complete graph is stable, while if $\alpha \geq 1$ then the star is stable.

Proof:

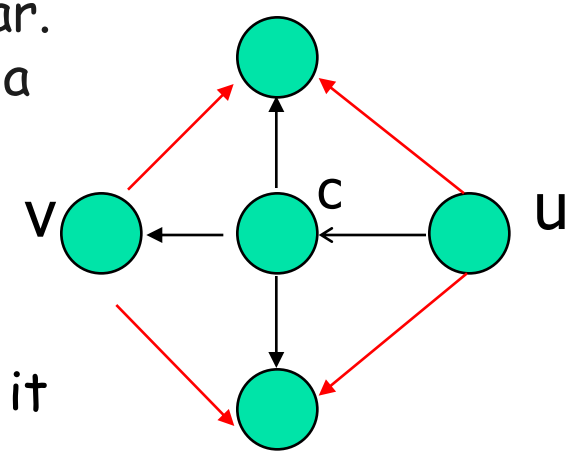
$\alpha \leq 1$

By definition, we have to find a NE S inducing a clique. Actually, any arbitrary strategy profile S inducing a clique is a NE. Indeed, if a node removes any $k \geq 1$ owned edges, it saves αk in the building cost, but it pays $k \geq \alpha k$ more in the usage cost (the k detached nodes are now at distance 2)



Proof (continued) $\alpha \geq 1$

By definition, we have to find a NE S inducing a star. Actually, any arbitrary strategy profile S inducing a star is a NE. Indeed:



Center c cannot change its strategy, otherwise its cost increase to infinity

If a leaf v not buying edges buys any $1 \leq k \leq n-2$ edges it pays αk more in the building cost, but it saves only $k \leq \alpha k$ in the usage cost

For a leaf u buying an edge, its cost is $\alpha+1+2(n-2)$ and we have two cases:

Case 1: u maintains (u,c) and buys any $1 \leq k \leq n-2$ additional edges; this case is similar to the previous one.

Case 2: u removes (u,c) and buys any $1 \leq k \leq n-2$ edges; thus, it pays αk in the building cost, and its usage cost becomes $k+2+3(n-k-2)$, and so its total cost becomes:

distance to adjacent nodes — distance to c — distance to other nodes

$$\alpha k+k+2+3n-3k-6 = \alpha+[\alpha(k-1)-2k+n]+2(n-2) \geq$$

$$\alpha+[k-1-2k+n]+2(n-2) = \alpha+[n-k-1]+2(n-2)$$

which is at least equal to the initial cost of $\alpha+1+2(n-2)$, since the quantity in square brackets is at least 1, being $1 \leq k \leq n-2$. ■

Theorem 2

For $\alpha \le 1$ and $\alpha \ge 2$ the PoS is 1. For $1<\alpha<2$ the PoS is at most 4/3

Proof: From Lemma 1 and 2, for $\alpha \le 1$ (respectively, $\alpha \ge 2$) a complete graph (respectively, a star) is both optimal and stable, and so the claim follows.

$1<\alpha<2$ K_n is an optimal solution (Lemma 1), and a star T is stable (Lemma 2); then

$$PoS \le \frac{SC(T)}{SC(K_n)} = \frac{\alpha(n-1) + 2(n-1)^2}{\alpha n(n-1)/2 + n(n-1)} < \frac{2n(n-1)}{n(n-1)/2 + n(n-1)} = 4/3$$

$\alpha(n-1) < 2(n-1)$ for $1<\alpha<2$

$\alpha n(n-1)/2 + n(n-1) > n(n-1)/2+n(n-1)$ for $1<\alpha<2$

■

What about the Price of Anarchy?

…for $\alpha<1$ the complete graph is the only stable network,
(try to prove that formally)
hence PoA=1…

…for larger value of α?

State-of-the-art

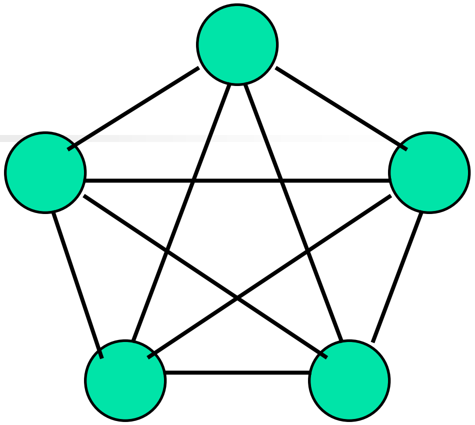
$\alpha = 0$	1	2	$\sqrt[3]{n/2}$	$\sqrt{n/2}$	$O(n^{1-\epsilon})$	$273n$	$12n \lg n$	∞
PoA	1	$\leq \frac{4}{3}$	≤ 4	≤ 6	$\Theta(1)$	$2^{O(\sqrt{\log n})}$	< 5	≤ 1.5

Many of these results are quite technical; we will show a simpler bound, namely that

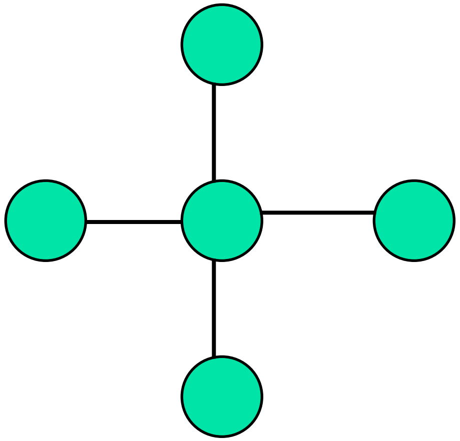
PoA=O(√α)

Some more notation

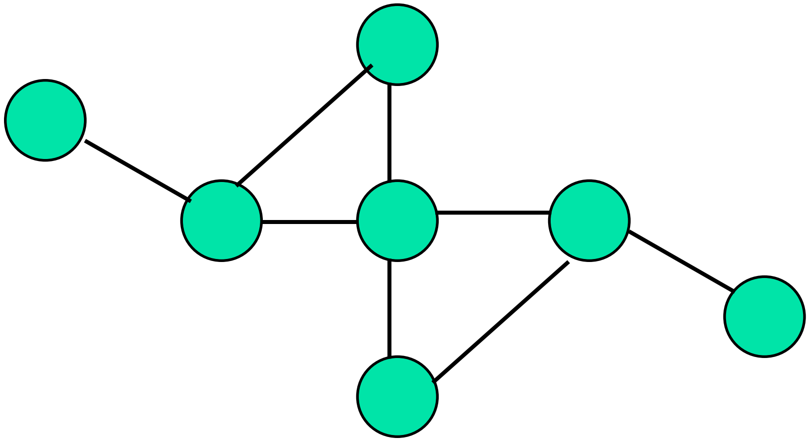
The diameter of a graph G is the maximum distance between any two nodes



diam=1



diam=2

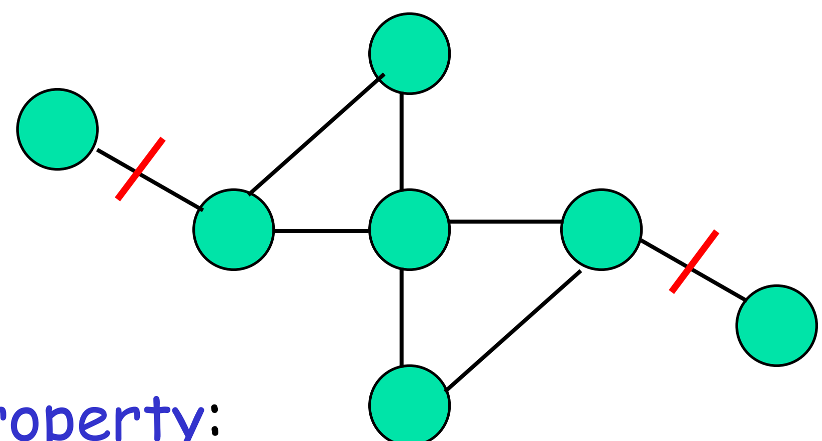


diam=4

Some more notation

An edge e is a cut edge (a.k.a. bridge) of a graph G=(V,E) if G-e is disconnected

G-e=(V,E\{e})



A simple property:

Any graph has at most n-1 cut edges (indeed, if we take any spanning tree T of G, all the non-tree edges cannot be bridges, and T has exactly n-1 edges (not all of them are bridges, clearly))

Theorem 3

The PoA of the LCG is at most $6\sqrt{\alpha} + 3$.

proof

It follows from the following lemmas:

Lemma 3

The diameter of any stable network is at most $2\sqrt{\alpha} + 1$.

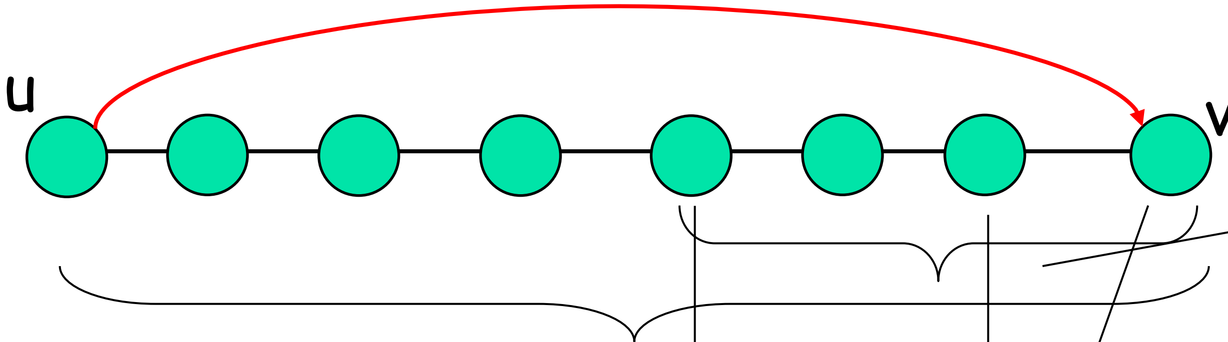
Lemma 4

The SC of any stable network with diameter d is at most $3d$ times the optimum SC.

proof of Lemma 3

G: stable network

Consider a shortest path in G between two nodes u and v



k vertices reduce their distance from u

$2k \leq dist_G(u,v) \leq 2k+1$
for some k

from $\geq 2k$ to 1 → $\geq 2k-1$

from $\geq 2k-1$ to 2 → $\geq 2k-3$

⋮

from $\geq k+1$ to k → ≥ 1

$$\sum_{i=0}^{k-1}(2i+1)=k^2$$

…since G is stable:

⇒ $\alpha \geq k^2$ ⇒ $k \leq \sqrt{\alpha}$

⇒ $dist_G(u,v) \leq 2\sqrt{\alpha} + 1$

To prove Lemma 4 we will make use of the following:

Proposition 1

Let G be a network with diameter d, and let e=(u,v) be a non-cut edge. Then in G-e, every node w increases its distance from u by at most 2d

Proposition 2

Let G be a stable network, and let F be the set of non-cut edges bought by a node u. Then $|F| \leq (n-1)2d/\alpha$

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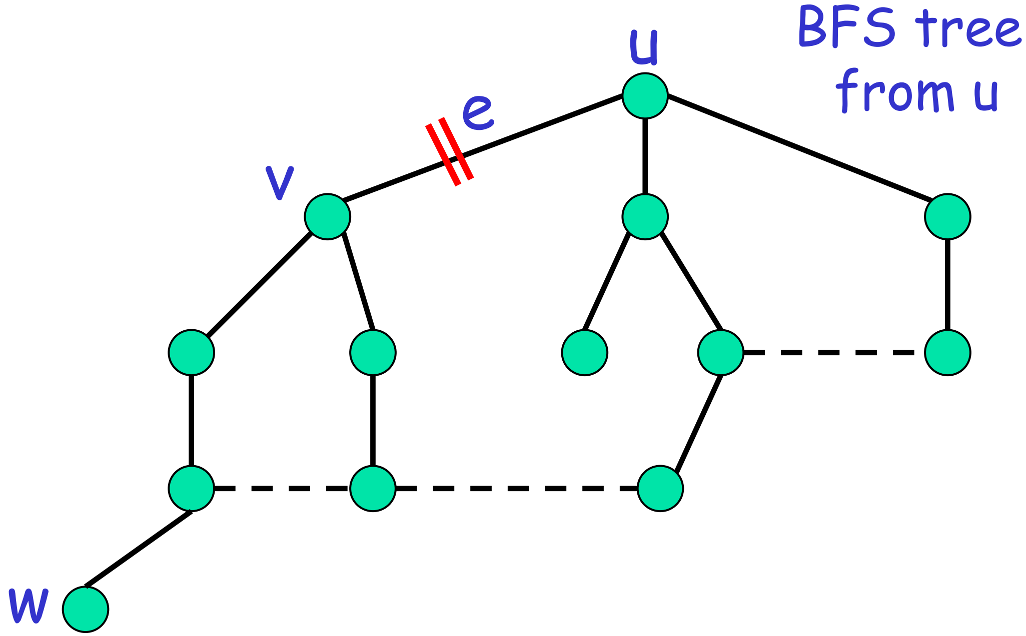

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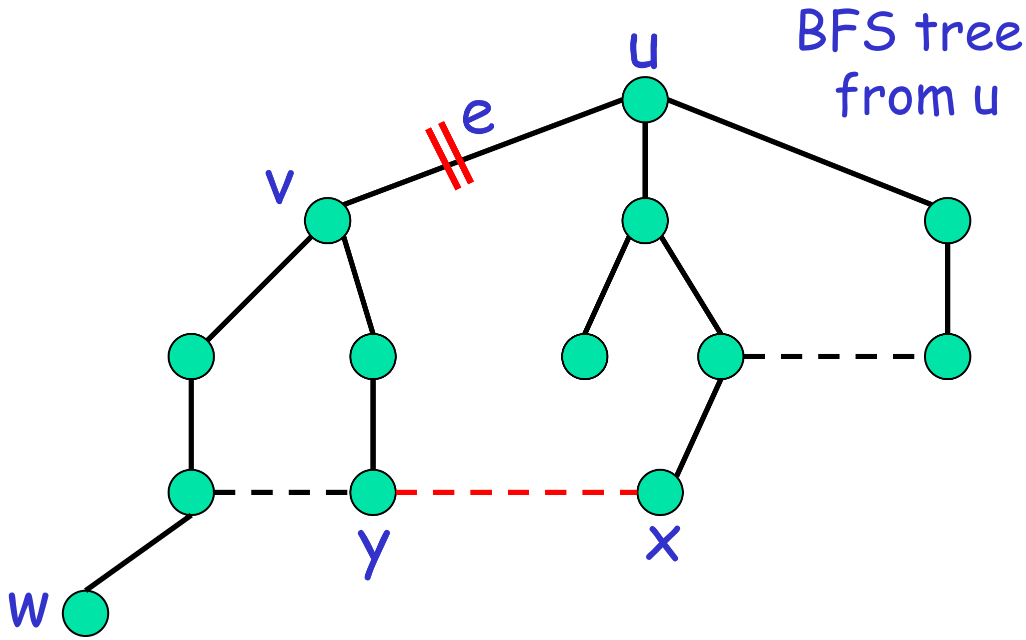

(x,y):
any edge crossing
the cut induced
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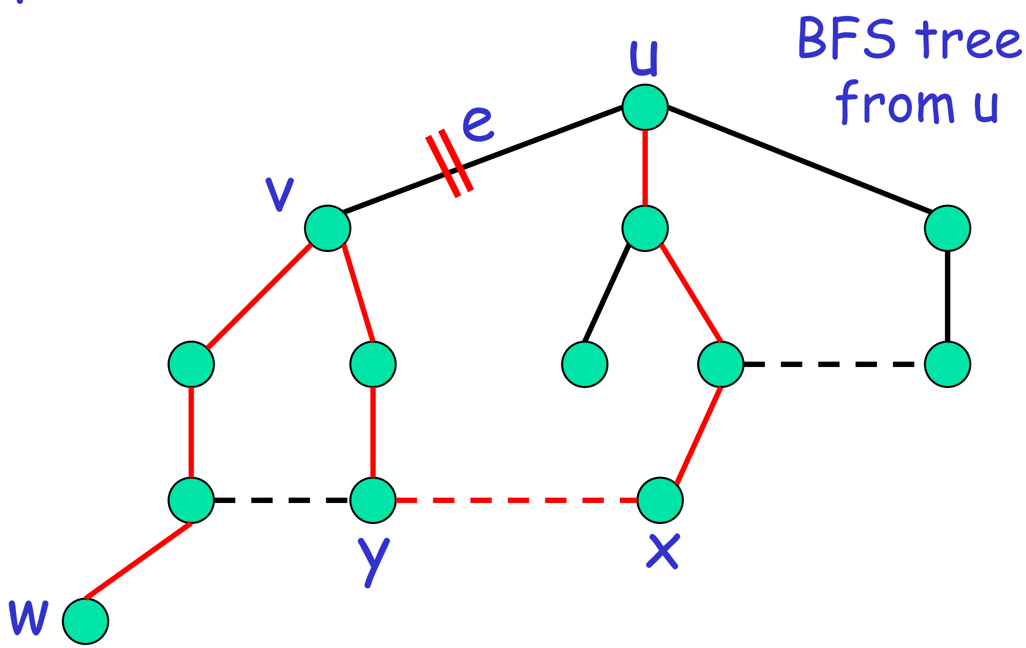


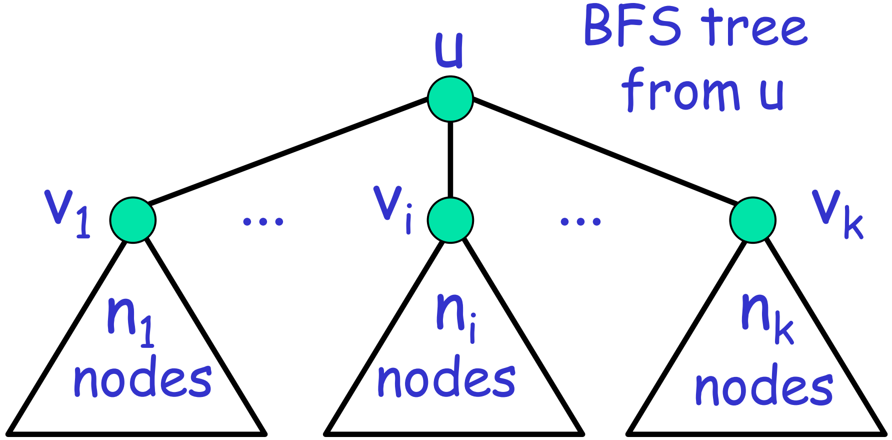
(x,y): any edge crossing the cut induced by the removal of e

$$d_{G-e}(u,w) \le \underbrace{d_G(u,x)}_{\le d} + 1 + \underbrace{d_G(y,v)}_{\le d} + \underbrace{d_G(v,w)}_{= d_G(u,w)-1} \le d_G(u,w) + 2d$$

Proposition 2

Let G be a stable network, and let F be the set of non-cut edges bought by a node u. Then $|F| \leq (n-1)2d/\alpha$

proof



$k = |F|$

if u removes (u,v_i) saves α and its distance cost increases by at most $2d\, n_i$ (Prop. 1)

since G is stable:

$\alpha \leq 2d\, n_i$

by summing up for all i

$$k\,\alpha \leq 2d \sum_{i=1}^{k} n_i \leq 2d\,(n-1)$$

⇒

$$k \leq (n-1)\, 2d/\alpha$$

■

Lemma 4

The SC of any stable network $G=(V,E)$ with diameter d is at most $3d$ times the optimum SC.

proof

$OPT \geq \alpha\,(n-1) + n(n-1)$ [notice this is the building cost of a star and the usage cost of a clique!]

$$SC(G)= \underbrace{\sum_{u,v} d_G(u,v)}_{\leq d\,n(n-1)\ \leq d\ OPT} + \alpha\,|E| \leq d\ OPT + 2d\ OPT = 3d\ OPT$$

$$\alpha|E| = \alpha\underbrace{|E_{cut}|}_{\leq (n-1)} + \alpha\underbrace{|E_{non\text{-}cut}|}_{\leq n(n-1)2d/\alpha \text{ (Prop. 2)}} \leq \alpha(n-1)+n(n-1)2d \leq 2d\ OPT$$

■